Appendix for Probabilistic Graphical Models for Boosting 
Cardinal and Ordinal Peer Grading in MOOCs 

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Cardinal Model Inference 
In this section, we present details of the inference procedures 
for the cardinal models (PG4 and PG5). Inference is analogous 
to Gibbs sampling but strictly speaking it is not, due to 
the lack of closed form for some distributions. In what follows, 
we use the same notational convention as in the paper. 
In addition, Z denotes the set of all peer assessments and 
v → u means that v grades the submission by u. 
The joint distribution for both PG4 and PG5 is: 
P(Z, (s_u)_{u ∈ U}, (b_v)_{v ∈ V}, (τ_v)_{v ∈ V}) = \prod_{u ∈ U} (s_u | μ_u, γ_u, α_u) \prod_{v ∈ V} (τ_v | s_u, b_v, τ_v) \prod_{u ∈ Z} (z_u | s_u, b_v, τ_v) 

We derive the posterior distributions for the latent 
variables of PG4. For a particular student u_i, we derive the 
posterior distribution of her latent true score as below (MB 
denotes the Markov blanket). As we can see, the posterior 
distribution has no closed form and so we have to use discrete 
approximation with fine separating intervals (intervals of width 0.1 from 0 to the full score of the assignment).

s_i ∼ P(s_i | MB(s_i)) 
∝ p(s_i | μ, γ, α) p(τ_i | s_i, b_i, τ_i) \prod_{v ∈ V \cup u_i} p(z_{u_j} | s_{u_j}, b_{u_j}, τ_{u_j}) 
∝ G_{u_i} \exp \left(-\frac{1}{2} \left(γ_0 s_i - μ_0 \right)^2 - β_0 τ_i + \sum_{v ≠ u_i} τ_v (z_{u_i} - (s_{u_i} + b_{u_i}))^2 \right) 

τ_i ∼ P(τ_i | MB(τ_i)) ∝ P(τ_i | z_{u_i}, b_i, s_i, s_{u_i}, α) 
∝ p(τ_i | s_i, b_i) \prod_{u, v ≠ u_i} p(z_{u_j} | s_{u_j}, b_{u_j}, τ_{u_j}) 
∝ τ_{u_i}^{λ - 1} \exp \left(-β_0 τ_i + \frac{1}{2} \sum_{u, v ≠ u_i} (log τ_v - log 2π - (z_{u_i} - (s_i + b_i))^2) \right) 

where : R_{u_i} = γ_0 + \sum_{v ≠ u_i} τ_v, Y_s = γ_0 μ_0 + \sum_{v ≠ u_i} τ_v (z_{u_i} - b_i), 
G_{u_i} = \frac{β_0 e^{-2\frac{Y_s}{R_{u_i}}}}{Γ(\frac{λ}{2})} \frac{1}{R_{u_i}^{\frac{λ}{2}}} 

For a particular peer grader v_i, we derive the closed-form posterior 
distribution for her bias as follows: 
b_i ∼ P(b_i | MB(b_i)) ∝ P(b_i | z_{u_i}, τ_i, s_i, s_0, α) 
∝ p(b_i | s_0) \prod_{u ⊆ V \cup u_i} p(z_{u_j} | s_{u_j}, b_{u_j}, τ_{u_j}) 
∝ \exp \left(-\frac{1}{2} \left(γ_0 b_i - 2(\sum_{u_i = u} τ_i (z_{u_i} - s_i)) \right)^2 \right) 

Similarly, we can derive the closed-form posterior distribution 
for her reliability as follows: 
τ_i ∼ P(τ_i | MB(τ_i)) ∝ P(τ_i | z_{u_i}, b_i, s_i, s_{u_i}, β) 
∝ p(τ_i | s_i, b_i, τ_i) \prod_{u, v ≠ u_i} p(z_{u_j} | s_{u_j}, b_{u_j}, τ_{u_j}) 
∝ τ_{u_i}^{n_{u_i} - 1} \exp \left(-β_0 τ_i + \frac{1}{2} \sum_{u, v ≠ u_i} (z_{u_i} - (s_i + b_i))^2 \right) 

It is in the form of a gamma distribution: 
τ_i ∼ \mathcal{G} \left(\frac{n_{u_i} + n_{v_i}}{2}, \frac{n_{u_i} + n_{v_i}}{2} \right) 

By taking the same approach, we can derive the correspon- 
ding distributions for PG5: 
s_i ∼ \mathcal{N} \left(γ_0 μ_0 + β_0 τ_0 + \sum_{u, v ≠ u_i} \frac{τ_v}{\lambda} (z_{u_i} - b_i), \frac{1}{γ_0 + β_0 + \sum_{u, v ≠ u_i} \frac{τ_v}{\lambda}} \right) 
b_i ∼ \mathcal{N} \left(\sum_{u, v ≠ u_i} \frac{τ_v}{\lambda} (z_{u_i} - s_i), \frac{1}{n_{u_i} + n_{v_i} \frac{τ_v}{\lambda}} \right) 
τ_i ∼ P(τ_i | MB(τ_i)) 
∝ \tau_{u_i}^{n_{u_i} - 1} \exp \left(-β_0 τ_i + \frac{1}{2} \sum_{u_i = u} (z_{u_i} - (s_i + b_i))^2 \right) 

During the sampling procedure, we run for 300 iterations 
with the first 60 samples removed as burn-in.
Parameter Sensitivity of Cardinal Models

As in the paper, we mainly tune $\beta_0$ in $PG_4$ and $\lambda$ in $PG_5$. There is an optimal $\beta_0$ for $PG_4$ and $\lambda$ for $PG_5$ for each specific setting of $\eta_0$ and $\gamma_0$, as illustrated for $PG_5$ in Figure 1. We also note that the error bar generally gets wider as $\lambda$ increases. This indicates that the confidence of the predicted grades gets lower, in line with the posterior distribution derived for the true score. Similar observations have been noted for other settings of $\eta_0$ and $\gamma_0$ as well but they are not included due to the page limit.

For other hyperparameters, we have observed that $\eta_0$ and $\gamma_0$ are not as sensitive as $\beta_0$ in $PG_4$ and $\lambda$ in $PG_5$. Namely, as long as $\eta_0$ and $\gamma_0$ in a reasonable range, we could obtain comparable accuracy by tuning $\beta_0$ or $\lambda$. The results in Table 3 and Table 6 of the paper were obtained after searching through combinations of the hyperparameters.

Ordinal Model Inference

This section describes the inference procedures for the pure cardinal models and the hyperparameter ranges considered, which are also applied to the “Cardinal + Ordinal” models. Basically, we use gradient descent with a learning rate of $1/\sqrt{t}$ at iteration $t$, stochastic gradient descent for the Bradley-Terry model and BT+G, and alternating block coordinate descent for RBTL.

Bradley-Terry Model

The cost function for the Bradley-Terry model is:

$$ L = \lambda_u a^2 + \frac{\lambda}{2\sigma^2} \sum_{u \in U} (s_u - \mu)^2 - \sum_{v \in V} \sum_{u_i > v} \sum_{j \in \rho(v)} s_{u_j} \log(\text{hypothesis}) $$

The update equations for the true score ($\sigma = 1$) are:

$$ \Delta s_u = -\eta \left( \frac{\lambda(s_u - \mu)}{2\sigma^2} - a(s_{u_i} - s_u) \right)(1 - \text{hypothesis}) $$

$$ \Delta s_{u_j} = -\eta \left( \frac{\lambda(s_{u_j} - \mu)}{2\sigma^2} - v(s_{u_j} - s_u) \right)(1 - \text{hypothesis}) $$

We fix $b = 0.2$, and tune $\lambda$ in the range $[0.1, 2]$ with intervals of 0.2 and $\lambda_u$ in $[50, 300]$ with intervals of 50.

BT+G Model

The cost function for BT+G is:

$$ L = \frac{\lambda_v}{2\sigma^2} (\tau_v - 1)^2 + \frac{\lambda}{2\sigma^2} \sum_{u \in U} (s_u - \mu)^2 - \sum_{v \in V} \sum_{u_i > v} \sum_{j \in \rho(v)} s_{u_j} \log(\text{hypothesis}) $$

The update equations for the true score ($\sigma = 1, \tau = 1$) are:

$$ \Delta \tau_v = -\eta \left( \frac{\lambda_v (\tau_v - 1)}{2\sigma^2} - (s_{u_i} - s_u) \right)(1 - \text{hypothesis}) $$

$$ \Delta s_{u_i} = -\eta \left( \frac{\lambda(s_u - \mu)}{2\sigma^2} - \tau_v \right)(1 - \text{hypothesis}) $$

$$ \Delta s_{u_j} = -\eta \left( \frac{\lambda(s_u - \mu)}{2\sigma^2} + \tau_v \right)(1 - \text{hypothesis}) $$

We tune $\lambda$ in $[0.1, 2]$ with intervals of 0.2 and $\lambda_v$ in $[0.5, 3]$ with intervals of 0.5.

For the “Cardinal+Ordinal” models, we use the same inference procedures and hyperparameter ranges for the above ordinal models, except that the parameter $\mu$ in the cost functions above are replaced by the predicted score $\mu_u$ for submission $u$ obtained from a cardinal model.